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Levinson's theorem for the Klein-Gordon equation in one dimension

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Abstract. In terms of the modified Sturm-Liouville theorem, the Levinson theorem for the one-dimensional Klein-Gordon equation with a symmetric potential $V(x)$ is established. It is shown that the number N_+ (N_-) of bound states with even (odd) parity is related to the phase shift $\eta_+(\pm M)[\eta_-(\pm M)]$ of the scattering states with the same parity at zero momentum as

$$\eta_+(M) - \eta_+(-M) = \begin{cases} (N_+ - 1/2)\,\pi & \text{for the non-critical case} \\ N_+\pi & \text{for the critical case } E = \pm M \end{cases}$$

and

$$\eta_-(M) - \eta_-(-M) = \begin{cases} N_-\pi & \text{for the non-critical case} \\ (N_- + 1/2)\,\pi & \text{for the critical case } E = \pm M. \end{cases}$$

The solution of the one-dimensional Klein-Gordon equation with the energy M or $-M$ is called as a half bound state if it is finite but does not decay fast enough at infinity to be square integrable.

PACS. 03.65.Ge Solutions of wave equations: bound states – 11.80.-m Relativistic scattering theory – 73.50.Bk General theory, scattering mechanisms

1 Introduction

The Levinson theorem [1], an important theorem in quantum scattering theory, established the relation between the total number of bound states and the phase shift at zero momentum. During the past half century, the Levinson theorem has been proved by several authors with different methods, and generalized to different fields [2–23].

The Klein-Gordon equation, which describes the motion of a relativistic scalar particle, is a second-order differential equation with respect to both space and time. When there exists a potential as the time component of a vector field, the eigenvalues for the Klein-Gordon equation are not necessary to be real and the eigenfunction satisfy the orthogonal relations with a weight factor [38,39] such that a parameter ϵ, which is not always real, appears in the normalized relation with a weight factor. As pointed out by Pauli and Snyder *et al.* [38,39], after Bose quantization those amplitudes with real and positive ϵ describe particles, and those with real and negative ϵ antiparticles.

Recall that in the three-dimensional spaces, two methods were used to set up the Levinson theorem for the Klein-Gordon equation. One was relied on the Green function method [5], where some formulas are valid only for the cases without complex energies. The other was based on a modified Sturm-Liouville theorem [9], by which the Levinson theorem for the Klein-Gordon equation was established for the cases even with complex energies. Furthermore, the Levinson theorem for the two-dimensional Klein-Gordon equation has been established by the Sturm-Liouville theorem [27].

With the wide interest in lower-dimensional field theory recently, it is worthwhile to study the Levinson theorem for the one-dimensional Klein-Gordon equation besides the study both in two dimensions and in three dimensions for completeness. Besides, it is found that the direct or implicit study of the one-dimensional Levinson theorem for the Schrödinger equation [16,28–37] has attracted much more attention than that of the two-dimensional and three-dimensional spaces. Moreover, the Levinson theorem for the one-dimensional Dirac equation has been established by the Green function method [23]. Therefore, we now attempt to set up the Levinson theorem for the Klein-Gordon equation in one dimension, which is the purpose of this paper.

This paper is organized as follows. In Section 2, we first review the properties of the Klein-Gordon equation,

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especially those related with the parameter ϵ. In Section 3, it is proved that the difference between the number of bound states of particle and that of antiparticle relies only on the changes of the logarithmic derivatives of the wave functions at $E = \pm M$ as the potential $V(x)$ changes from zero to the given value. In Section 4, it is turned out that these changes are associated with the phase shifts at $E = \pm M$ which then leads to the establishment of the Levinson theorem for the one-dimensional Klein-Gordon equation.

2 The Klein-Gordon equation

Throughout this paper the natural units $\hbar = c = 1$ are employed. Consider a relativistic scalar particle satisfying the Klein-Gordon equation

$$\left(-\nabla^2 + M^2\right)\psi(x) = [E - V(x)]^2\psi(x) \quad V(-x) = V(x), \tag{1}$$

where M and E denote the mass and the energy of the particle, respectively. For simplicity, we first discuss the case with a cutoff potential

$$V(x) = 0, \quad \text{when} \quad x \geq x_0, \tag{2}$$

where x_0 is a sufficiently large distance. Following the method in [24–26], the conclusions also hold if the potential vanishes faster than x^{-2} at infinity. Introduce a parameter λ for the potential $V(x)$

$$V(x,\lambda) = \lambda V(x), \tag{3}$$

where the potential $V(x,\lambda)$ changes from zero to the given potential $V(x)$ as λ increases from zero to one. On introducing the parameter λ, the one-dimensional Klein-Gordon equation can be modified as

$$\frac{\partial^2}{\partial x^2}\psi_E(x,\lambda) + [(E - V(x,\lambda))^2 - M^2]\psi_E(x,\lambda) = 0. \tag{4}$$

For the symmetric potential, the eigenfunction can be combined into those with a definite parity, which satisfies the following boundary conditions in the origin

$$\begin{aligned} \psi_E^{(\mathrm{o})}(x,\lambda)\Big|_{x=0} &= 0 \quad \text{for the odd-parity case} \\ \left.\frac{\partial \psi_E^{(\mathrm{e})}(x,\lambda)}{\partial x}\right|_{x=0} &= 0 \quad \text{for the even-parity case.} \end{aligned} \tag{5}$$

We therefore only need to discuss the wave function in the range $[0,\infty)$ with the given parities, even parity case and odd parity one, respectively.

Denote by $\psi^*_{\overline{E}}(x,\lambda)$ the solution of equation (4) corresponding to the energy $\overline{E}$

$$\frac{\partial^2}{\partial x^2}\psi^*_{\overline{E}}(x,\lambda) + [(\overline{E} - V(x,\lambda))^2 - M^2]\psi^*_{\overline{E}}(x,\lambda) = 0. \tag{6}$$

Multiplying equations (4, 6) by $\psi^*_{\overline{E}}(x,\lambda)$ and $\psi_E(x,\lambda)$, respectively, and calculating their difference, we obtain

$$\begin{aligned} \frac{\partial}{\partial x}\left\{\psi_E(x,\lambda)\frac{\partial \psi^*_{\overline{E}}(x,\lambda)}{\partial x} - \psi^*_{\overline{E}}(x,\lambda)\frac{\partial \psi_E(x,\lambda)}{\partial x}\right\} \\ = -(\overline{E}^* - E)\psi^*_{\overline{E}}(x,\lambda)(\overline{E}^* + E - 2V(x,\lambda))\psi_E(x,\lambda). \end{aligned} \tag{7}$$

It is well-known [38,39] that, due to the so-called Klein paradox, the eigenvalues are not necessarily real for some potential $V(x)$. Integrating equation (7) over the whole space and noting that $\psi_E(x,\lambda)\psi'^*_{\overline{E}}(x,\lambda) - \psi^*_{\overline{E}}(x,\lambda)\psi'_E(x,\lambda)$ vanishes both in the origin and at infinity for the physically acceptable solutions with the different energies E and $\overline{E}$, we obtain the weighted orthogonality relation for the radial function

$$\begin{aligned} (\overline{E}^* - E)\int_0^\infty \psi^*_{\overline{E}}(x,\lambda)(\overline{E}^* + E - 2V(x,\lambda)) \\ \times\, \psi_E(x,\lambda)\mathrm{d}x = 0. \end{aligned} \tag{8}$$

Actually, we can always obtain the *real* solutions for the *real* energies. It is easy to find from equation (8) that the normalized relation for the solutions with real energies are not always positive on account of the weight factor $(\overline{E} + E - 2V(x,\lambda))$

$$\begin{aligned} \int_0^\infty \psi_{\overline{E}}(x,\lambda)(\overline{E} + E - 2V(x,\lambda))\psi_E(x,\lambda)\mathrm{d}x \\ = \begin{cases} \epsilon_E\delta(\overline{E} - E)(E^2 - M^2)^{1/2}/|E| & \text{when } |E| > M \\ \epsilon_E\delta_{\overline{E}E} & \text{when } |E| < M, \end{cases} \end{aligned} \tag{9}$$

which implies

$$\epsilon_E = 2\int_0^\infty \psi_E^2(x,\lambda)[E - V(x)]\mathrm{d}x.$$

The parameter ϵ_E, which depends on the particular radial function $\psi_E(x,\lambda)$, may be either positive, or negative, or vanishing. Normalized factors of the solutions cannot change the sign of ϵ_E. Generally, if $\psi_E(x,\lambda)$ is a complex solution of equation (4) with a complex energy E, then $\psi^*_E(x,\lambda)$ is also a solution with a complex energy E^*, and a complex ϵ_E appears for a pair of complex solutions. It is evident after Bose quantization that those $\psi_E(x,\lambda)$ with positive ϵ_E describes particles and those with negative ϵ_E antiparticle. The solution with zero ϵ_E can be regarded as a pair of solutions with infinitesimal $\pm|\epsilon_E|$, which describe a pair of the particle and antiparticle bound states. The Hamiltonian and charge operator cannot be written as the diagonal forms for the solutions with complex ϵ_E, they therefore describe neither particles nor antiparticle. In the present paper, we only count the number of bound states with the real ϵ_E, which are named particle and antiparticle bound states, respectively.

Since we can always obtain the *real* solution for the *real* energy, we now solve equation (4) in two regions and match two real solutions at x_0. Similar to our previous work [24–29], the matching condition can be taken as

$$A(E,\lambda) \equiv \left\{\frac{1}{\psi_E(x,\lambda)}\frac{\partial\psi_E(x,\lambda)}{\partial x}\right\}_{x=x_0-} = \left\{\frac{1}{\psi_E(x,\lambda)}\frac{\partial\psi_E(x,\lambda)}{\partial x}\right\}_{x=x_0+} \equiv B(E). \quad (10)$$

According to the condition (5), there exists only one solution near the origin. For example, for the free particle ($\lambda = 0$), the solution of equation (4) in the range $[0, x_0]$ is real and read as

$$\psi_E^{(e)}(x,0) = \begin{cases} \cos(kx) & \text{when } |E| > M \text{ and } k = \sqrt{E^2 - M^2} \\ \cosh(\kappa x), & \text{when } |E| \le M \text{ and } \kappa = \sqrt{M^2 - E^2} \end{cases} \quad (11)$$

for the even-parity case, and

$$\psi_E^{(o)}(x,0) = \begin{cases} \sin(kx) & \text{when } |E| > M \text{ and } k = \sqrt{E^2 - M^2} \\ \sinh(\kappa x) & \text{when } |E| \le M \text{ and } \kappa = \sqrt{M^2 - E^2} \end{cases} \quad (12)$$

for the odd-parity case.

In the range $[x_0, \infty)$, we have $V(x) = 0$. For $|E| > M$, there exist two oscillatory solutions of equation (4) whose combination can always satisfy equation (10), so that there is a continuous spectrum for $|E| > M$. Assuming that the $\eta_\pm(k,\lambda)$ are zero for the free particles ($\lambda = 0$), we have

$$\psi_E(x,\lambda) = \begin{cases} \cos(kx + \eta_+(k,\lambda)) & \text{for the even-parity case} \\ \sin(kx + \eta_-(k,\lambda)) & \text{for the odd-parity case.} \end{cases} \quad (13)$$

$$\eta_\pm(k,0) = 0 \qquad \text{when } k > 0. \quad (14)$$

Some remarks will be given here. First, the wave function in equation (13) seems not to have a definite parity. In fact, the solutions (13) are only suitable in the region $[x_0, \infty)$. The corresponding solutions in the region $(-\infty, -x_0]$ can be calculated according to the parity of the solution. For example, for the odd-parity case, the solution in the region $(-\infty, -x_0]$ is

$$-\sin(k|x| + \eta_-(k,\lambda)) = \sin(kx - \eta_-(k,\lambda)).$$

Second, the solutions (13) for the even-parity case can be rewritten as

$$\sin(kx + \eta_+(k,\lambda) + \pi/2). \quad (15)$$

The $\eta_+(k,\lambda) + \pi/2$ plays the same role for the even-parity case as $\eta_-(k,\lambda)$ for the odd-parity case. We therefore only need to establish the Levinson theorem for the odd-parity case, and that for the even-parity case can be obtained through replacing $\eta_-(k,\lambda)$ by $\eta_+(k,\lambda) + \pi/2$.

Since there is only one finite solution at infinity for $|E| \le M$, both for even-parity case and for the odd-parity case

$$\psi_E(x,\lambda) = \exp(-\kappa x), \qquad \text{when } x_0 \le x < \infty. \quad (16)$$

The solution satisfying equation (10) will not always exist for $|E| \le M$. Except for $E = \pm M$, if and only if there exists a solution of energy E satisfying equation (10), a bound state appears at this energy, which means that there is a discrete spectrum for $|E| \le M$. The finite solution for $E = \pm M$ is a constant one. It decays not fast enough to be square integrable such that it is not a bound state if equation (10) is satisfied.

For the case with a *real* energy, integrating equation (7) in two regions $[0, x_0]$ and $[x_0, \infty)$, respectively, and taking the limit $\overline{E} \to E$, we obtain the following equations in terms of the boundary condition that $\psi_E(0) = 0$ and $\psi_E(\infty) = 0$ for $|E| < M$,

$$\frac{\partial A(E,\lambda)}{\partial E} \equiv \frac{\partial}{\partial E}\left(\frac{1}{\psi_E(x,\lambda)}\frac{\partial\psi_E(x,\lambda)}{\partial x}\right)_{x=x_0-} = -2\psi_E(x_0,\lambda)^{-2}\int_0^{x_0}\psi_E(x,\lambda)^2[E - V(x,\lambda)]\,\mathrm{d}x \quad (17a)$$

and

$$\frac{\mathrm{d}B(E)}{\mathrm{d}E} \equiv \frac{\mathrm{d}}{\mathrm{d}E}\left(\frac{1}{\psi_E(x)}\frac{\mathrm{d}\psi_E(x)}{\mathrm{d}x}\right)_{x=x_0+} = 2\psi_E(x_0)^{-2}\int_{x_0}^{\infty}\psi_E(x)^2E\,\mathrm{d}x. \quad (17b)$$

It is demonstrated from equation (17) that $A(E,\lambda)$ is no longer monotonic with respect to the energy, but $B(E)$ is still monotonic with respect to the energy if the energy does not change sign. However, their difference, $B(E) - A(E,\lambda)$, is monotonic with respect to the energy for the particle ($\epsilon_E > 0$)and for the antiparticle ($\epsilon_E < 0$), respectively

$$\frac{\partial}{\partial E}\{A(E,\lambda) - B(E)\} = -\psi_E(x_0,\lambda)^{-2}\,\epsilon_E. \quad (18)$$

Equation (18) is called the modified Sturm-Liouville theorem. It is owing to the modified Sturm-Liouville theorem that a bound state can be identified as a particle ($\epsilon_E > 0$) or an antiparticle one ($\epsilon_E < 0$) by whether $A(E,\lambda) - B(E)$ decreases or increases as the energy E increases. From equation (10), we have

$$\tan\eta_-(k,\lambda) = -\tan(kx_0)\,\frac{A(E,\lambda) - k\,\mathrm{cotan}(kx_0)}{A(E,\lambda) + k\tan(kx_0)}, \quad (19)$$

for the odd-parity case, and the similar formula for the even-parity case can be obtained by replacing $\eta_-(k,\lambda)$ with $\eta_+(k,\lambda) + \pi/2$.

The $\eta_-(E,\lambda)$ is determined from equation (19) up to a multiple of π due to the period of the tangent function. Following our previous papers [6,7,24–29], we use the convention for determining the phase shift absolutely that the $\eta_-(E,0)$ for the free particle ($\lambda = 0$) is defined to be zero,

$$\eta_-(E,0) = 0, \qquad \text{where} \quad \lambda = 0. \tag{20}$$

As shown in equation (9) that the scattering states, $|E| > M$, are normalized as the Dirac δ function, where the main contribution to the integration comes from the radial functions in the region far away from the origin. Therefore, we may change the integral region in equation (9) to $[x_0, \infty)$ where there is no potential. Substituting equation (13) into equation (9), we obtain

$$\epsilon_E = \pi E, \qquad \text{when} \quad |E| > M. \tag{21}$$

All the scattering states with the positive energy ($E > M$) describe particles and those with the negative energy ($E < -M$) describe antiparticle.

3 The number of bound states

In our previous works, the Levinson theorems have been established by the Sturm-Liouville theorem, whose fundamental trick is the definition of a phase angle which is monotonic with respect to the energy [40]. However, due to the factor $(\overline{E}^* + E - 2V)$ in equation (9), the Sturm-Liouville theorem has to be modified for the the one-dimensional Klein-Gordon equation. Fortunately, as shown in equation (18), the difference of the logarithmic derivatives at two sides of x_0, $A(E,\lambda) - B(E)$, is monotonic with respect to the energy for the particle ($\epsilon_E > 0$) and for the antiparticle ($\epsilon_E < 0$), respectively. From equation (16), we get

$$B(E) = \left(\frac{1}{\psi_E(x,\lambda)}\frac{\partial \psi_E(x,\lambda)}{\partial x}\right)_{x=x_0+} \leq B(\pm M) = 0 \qquad \text{when} \quad |E| \leq M. \tag{22}$$

On the other hand, when $\lambda = 0$, the logarithmic derivative at $x = x_0-$ can be calculated from equations (11, 12) for $|E| \leq M$

$$A(E,0) = \left(\frac{1}{\psi_E(x,0)}\frac{\partial \psi_E(x,0)}{\partial x}\right)_{x=x_0-} = \kappa \tanh(\kappa x_0) \geq A(\pm M, 0) = 0 \qquad \text{when} \quad |E| \leq M \tag{23}$$

for the even-parity case, and

$$A(E,0) = \left(\frac{1}{\psi_E(x,0)}\frac{\partial \psi_E(x,0)}{\partial x}\right)_{x=x_0-} = \kappa \coth(\kappa x_0) \geq A(\pm M, 0) = x_0^{-1} \qquad \text{when} \quad |E| \leq M \tag{24}$$

for the odd-parity case.

The logarithmic derivative $B(E)$ does not depend on λ. It is evident to see from equations (22, 24) that there is no overlap between two variant ranges of two logarithmic derivatives for the odd-parity case, *i.e.* there is no bound state for the free particle for the odd-parity case. However, there is one point overlap from equations (22, 23). It means that there is a finite solution at $E = \pm M$ when $\lambda = 0$ in even-parity case. It is nothing but a constant solution. This solution is finite but does not decay fast enough at infinity to be square integrable. It is not a bound state but a half bound state which will be discussed in Section 4.

When λ changes from zero to the given potential, $B(E)$ does not change, but $A(E,\lambda)$ changes continuously except for the points where $\psi_E(x_0) = 0$ and $A(E,\lambda)$ tends to infinity. Generally, $A(E,\lambda)$ is continuous except for those finite points and intersects with the curve $B(E)$ several times for $|E| \leq M$. The bound state will appear only if a point of intersection occurs. The number of the points of intersection is nothing but the number of the bound states. It is shown from equation (18) that the relative slope with respect to the energy at the point of intersection decides whether the bound state describes a particle or an antiparticle.

Denote by $n_-^+(\lambda)$ the number of particle bound states and by $n_-^-(\lambda)$ the number of antiparticle bound states for the odd-parity case. Their difference is denoted by $N_-(\lambda)$

$$N_-(\lambda) = n_-^+(\lambda) - n_-^-(\lambda). \tag{25}$$

When the potential $V(x,\lambda)$ changes with λ, the change of the number of points of intersection, when $|E| \leq M$, is only from two sources. First, the points of intersection move inwards or outward at $E = \pm M$; second, the curve $A(E,\lambda)$ intersects with the curve $B(E)$ or departs from it through a tangent point. For the second source, according to the modified Sturm-Liouville theorem (18), a pair of the particle and antiparticle bound states will be created or annihilated simultaneously, but $N_-(\lambda)$ keeps invariant, *i.e.* the change of the the difference $N_-(\lambda)$ only depends on the point of intersection moves in or out at $E = \pm M$.

We now discuss the properties when a point of intersection moves in or out at $E = \pm M$. We first discuss the situation that λ increases across the critical value λ_1 where $A(M,\lambda_1) = B(M) = 0$. There are two cases at the critical value

$$\text{(i)} \quad \left.\frac{\partial^{n'}}{\partial E^{n'}} A(E,\lambda_1)\right|_{E=M} = \left.\frac{\partial^{n'}}{\partial E^{n'}} B(E)\right|_{E=M}, \qquad \text{where} \quad 0 \leq n' < n,$$

$$(-1)^n \left.\frac{\partial^{n}}{\partial E^{n}} A(E,\lambda_1)\right|_{E=M} > (-1)^n \left.\frac{\partial^{n}}{\partial E^{n}} B(E)\right|_{E=M},$$

$$\text{(ii)} \quad \left.\frac{\partial^{n'}}{\partial E^{n'}}A(E,\lambda_1)\right|_{E=M} = \left.\frac{\partial^{n'}}{\partial E^{n'}}B(E)\right|_{E=M},$$

$$\text{where} \quad 0 \le n' < n,$$

$$(-1)^n\left.\frac{\partial^{n}}{\partial E^{n}}A(E,\lambda_1)\right|_{E=M} < (-1)^n\left.\frac{\partial^{n}}{\partial E^{n}}B(E)\right|_{E=M},$$

where n is a positive integer. It means that, for the energy $E < M$ but very near M,

$$A(E,\lambda_1) > B(E), \qquad \text{for the case (i)}, \tag{26a}$$

$$A(E,\lambda_1) < B(E), \qquad \text{for the case (ii)}. \tag{26b}$$

If $A(M,\lambda)$ decreases as λ increases across the critical value λ_1, a point of intersection moves into $E < M$ for the case (i) and moves out from $E < M$ for the case (ii), and simultaneously from equation (18), a scattering state with a positive energy becomes a particle bound state for the case (i) and an antiparticle bound state becomes a scattering state with a positive energy for the case (ii). For both cases $N_-(\lambda)$ increases by one. Conversely, if $A(M,\lambda)$ increases as λ increases across the critical value λ_1, $N_-(\lambda)$ decreases by one for both cases.

Second, we discuss the situation that λ increases across the critical value λ_2 where $A(-M,\lambda_2) = B(-M) = 0$. There are also two cases at the critical value

$$\text{(i)} \quad \left.\frac{\partial^{n'}}{\partial E^{n'}}A(E,\lambda_2)\right|_{E=-M} = \left.\frac{\partial^{n'}}{\partial E^{n'}}B(E)\right|_{E=-M},$$

$$\text{where} \quad 0 \le n' < n,$$

$$\left.\frac{\partial^{n}}{\partial E^{n}}A(E,\lambda_2)\right|_{E=-M} > \left.\frac{\partial^{n}}{\partial E^{n}}B(E)\right|_{E=-M}.$$

$$\text{(ii)} \quad \left.\frac{\partial^{n'}}{\partial E^{n'}}A(E,\lambda_2)\right|_{E=-M} = \left.\frac{\partial^{n'}}{\partial E^{n'}}B(E)\right|_{E=-M},$$

$$\text{where} \quad 0 \le n' < n,$$

$$\left.\frac{\partial^{n}}{\partial E^{n}}A(E,\lambda_2)\right|_{E=-M} < \left.\frac{\partial^{n}}{\partial E^{n}}B(E)\right|_{E=M}.$$

It means that, for the energy $E > -M$ but very near $-M$,

$$A(E,\lambda_2) > B(E), \qquad \text{for the case (i)}, \tag{27a}$$

$$A(E,\lambda_2) < B(E), \qquad \text{for the case (ii)}. \tag{27b}$$

If $A(-M,\lambda)$ decreases as λ increases across the critical value λ_2, a point of intersection moves in to $E > -M$ for the case (i) and moves out from $E > -M$ for the case (ii), and simultaneously from equation (18), a scattering state with a negative energy becomes an antiparticle bound state for the case (i) and a particle bound state becomes a scattering state with a negative energy for the case (ii). For both cases $N_-(\lambda)$ decreases by one. Conversely, if $A(M,\lambda)$ increases as λ increases across the critical value λ_2, $N_-(\lambda)$ increases by one for both cases.

Now, when λ increases from zero to one, we denote by $n(\pm M)$ the times that $A(\pm M,\lambda)$ decreases across the value $B(\pm M) = 0$, subtracted by the times that $A(\pm M,\lambda)$ increases across that value. Hence, we have

$$N \equiv N_-(1) = n(M) - n(-M). \tag{28}$$

Recall that from equation (25) $N_-(\lambda)$ is the difference between the numbers of particle and antiparticle bound states

$$N = N_-(1) = n_-^+(1) - n_-^-(1) \equiv n_-^+ - n_-^-. \tag{29}$$

4 The phase shifts

We now turn to the scattering states. The solutions in the region $[x_0,\infty)$ for the scattering states have been given by equation (13). The $\eta(\pm M,\lambda)$ is the limit of the $\eta(E,\lambda)$ as k tends to zero. Actually, what we are interested in is the $\eta(E,\lambda)$ at a sufficiently small momentum k, $k \ll 1/x_0$. Through equation (10), the $\eta(E,\lambda)$ at a sufficiently small momentum k can be calculated by equation (19) and the convention (20).

First, we obtain from equation (19) that

$$\left.\frac{\partial \eta_-(k,\lambda)}{\partial A(E,\lambda)}\right|_k = \frac{-k\cos^2\eta_-(k,\lambda)}{\{A\cos(kx) + k\sin(kx)\}^2} \le 0, \tag{30}$$

which is shown that the $\eta(E,\lambda)$ at a sufficiently small momentum k is monotonic with respect to the logarithmic derivative $A(E,\lambda)$ as λ increases.

Second, we discuss the non-critical case where

$$A(\pm M,1) \ne B(\pm M) = 0. \tag{31}$$

For the small momentum ($k \sim 0$) we obtain from equation (19)

$$\tan\eta_-(k,\lambda) \sim -(kx_0)\ \frac{A(0,\lambda) - c^2k^2 - x_0^{-1} + k^2x_0/3}{A(0,\lambda) - c^2k^2 + k^2x_0}, \tag{32}$$

where the expansion of $A(E,\lambda)$ for the small k is used

$$A(E,\lambda) \sim A(0,\lambda) - c^2k^2, \qquad c^2 \ge 0, \tag{33}$$

which is calculated from equation (17a). In both the numerator and the denominator of equation (32) we included the next leading term, which is only useful for the critical cases where the leading terms are canceled each other.

First, it can be seen from equation (32) that, except for $A(0,\lambda) = 0$, $\tan\eta_-(k,\lambda)$ tends to zero as k goes to zero, namely, $\eta_-(0,\lambda)$ is always equal to the multiple of π except for $A(0,\lambda) = 0$. When $A(0,\lambda) = 0$, the limit $\eta_-(0,\lambda)$ of the $\eta_-(k,\lambda)$ is equal to $(n+1/2)\pi$. It is not important for our discussion except for $A(0,1)=0$, which is called as the critical case and will be discussed later.

Second, for a sufficiently small k, if $A(E,\lambda)$ decreases as λ increases, $\eta_-(k,\lambda)$ increases monotonically. Assume that in the variant process $A(E,\lambda)$ may decreases through the value zero, but does not stop at this value. As $A(E,\lambda)$ decreases, each times $\tan\eta_-(k,\lambda)$ for the sufficiently small k changes sign from positive to negative, $\eta_-(0,\lambda)$ jumps by π. However, each times $\tan\eta_-(k,\lambda)$ changes sign from negative to positive, $\eta_-(0,\lambda)$ remains invariant. Conversely, if $A(E,\lambda)$ increases as λ increases, $\eta_-(k,\lambda)$ decreases monotonically. As $A(E,\lambda)$ increases, each time $\tan\eta_-(k,\lambda)$ changes sign from negative to positive, $\eta_-(0,\lambda)$ jumps by $-\pi$, and each time $\tan\eta_-(k,\lambda)$ changes sign from positive to negative, $\eta_-(0,\lambda)$ remains invariant.

Third, as λ increases from zero to one, $V(x,\lambda)$ changes from zero to the given potential $V(x)$ continuously. Each time $A(0,\lambda)$ decreases from near and larger than the value zero to smaller than that value, the denominator in equation (32) changes sign from positive to negative and the remaining factor remains positive, such that the $\eta_-(0,\lambda)$ jumps by π. Conversely, each time $A(0,\lambda)$ increases across the value zero, the $\eta_-(0,\lambda)$ jumps by $-\pi$. Each time $A(0,\lambda)$ decreases from near and larger than the value x_0^{-1} to smaller than that value, the numerator in equation (32) changes sign from positive to negative, but the remaining factor remains negative, such that the at zero momentum $\eta_-(0,\lambda)$ does not jump. Conversely, each time $A(0,\lambda)$ increases across the value x_0^{-1}, the $\eta_-(0,\lambda)$ does not jump, either.

Therefore, the $\eta_-(0)/\pi$ is just equal to the times $A(0,\lambda)$ decreases across the value zero as λ increases from zero to one, subtracted by the times $A(0,\lambda)$ increases across that value. As discussed in the previous section, we have proved that the difference of the two times is nothing but the number of bound states N_-, *i.e.*, for the non-critical cases, the Levinson theorem for the one-dimensional Klein-Gordon equation for the odd-parity case is written as

$$\eta_-(M)-\eta_-(-M)=N_-\pi. \tag{34}$$

Fourth, we now turn to discuss the critical case when $\lambda=1$,

$$A(\pm M,1)=B(\pm M)=0, \tag{35}$$

for the critical case, the constant solution $\psi_E(x)=c$ $(c\neq 0)$ in the range $[x_0,\infty)$ for zero energy will match this $A(0,1)$ at x_0. For the critical case, it is obvious that there exists a half-bound state both for the even-parity case and for the odd-parity case. A half-bound state is not a bound state, because its wave function is finite but not square integrable. As λ increases from a number near and less than one and finally reaches one, if the logarithmic derivative $A(0,\lambda)$ decreases and finally reaches, but not across, the value zero, according to the discussion in the previous section, a scattering state becomes a half bound state when $\lambda=1$. On the other hand, the denominator in equation (32) is proportional to k^2 such that $\tan\eta_-(k,1)$ tends to infinity, *i.e.* the $\eta_-(0,1)$ jumps by $\pi/2$. Therefore, for the critical case the Levinson theorem for the one-dimensional Klein-Gordon equation can be read as

$$\eta_-(+M)-\eta_-(-M)=(N_-+1/2)\pi. \tag{36}$$

Conversely, as λ increases and reaches one, if the logarithmic derivative $A(0,\lambda)$ increases and finally reaches the value zero, a bound state becomes a half bound state when $\lambda=1$, and the $\eta_-(\pm M,1)$ jumps by $-\pi/2$. In this case, the Levinson theorem (34) still holds.

At last, for the even-parity case, the only change is to replace $\eta_-(\pm M)$ by $\eta_+(\pm M)+\pi/2$. Similarly, denote by $n_+^+(\lambda)$ the number of particle bound states and by $n_+^-(\lambda)$ the number of antiparticle bound states for the even-parity case. Their difference is denoted by $N_+(\lambda)$. Therefore, the Levinson theorem for the one-dimensional Klein-Gordon equation for the even-parity case is

$$\eta_+(M)-\eta_+(-M)=(N_+-1/2)\pi, \quad \text{for the non-critical case,}$$

$$\eta_+(M)-\eta_+(-M)=N_+\pi, \quad \text{for the critical case.} \tag{37}$$

Note that for the free particle for the even-parity case, there is a half bound state at $E=\pm M$. It is the critical case where $\eta_+(\pm M)=0$ and $N_+=0$.

In summary, the Levinson theorem for the one-dimensional Klein-Gordon equation in one dimension can be written as

$$\eta_+(M)-\eta_+(-M)=\begin{cases}(N_+-1/2)\,\pi \\ \text{for the non-critical case} \\ N_+\pi \\ \text{for the critical case } E=\pm M\end{cases}$$

and

$$\eta_-(M)-\eta_-(-M)=\begin{cases}N_-\pi \\ \text{for the non-critical case} \\ (N_-+1/2)\,\pi \\ \text{for the critical case } E=\pm M.\end{cases} \tag{38}$$

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